

Boundary exponential stabilization of 1-D inhomogeneous quasilinear hyperbolic systems

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Abstract

This paper deals with the problem of boundary stabilization of first-order $n \times n$ inhomogeneous quasilinear hyperbolic systems. A backstepping method is developed. The main result supplements the previous works on how to design multi-boundary feedback controllers to realize exponential stability of the original nonlinear system in the spatial H^2 sense.

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1 Introduction and Main Result

Consider the following 1-D $n \times n$ inhomogeneous quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(x, u) \frac{\partial u}{\partial x} = F(x, u), \quad x \in [0, 1], \quad t \in [0, +\infty), \quad (1.1)$$

where, $u = (u_1, \dots, u_n)^T$ is an unknown vector function of (t, x) , $A(x, u)$ is an $n \times n$ matrix with C^2 entries $a_{ij}(x, u)(i, j = 1, \dots, n)$, $F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector valued function with C^2 components $f_i(x, u)(i = 1, \dots, n)$ with respect to u and

$$F(x, 0) \equiv 0. \quad (1.2)$$

Denote

$$\frac{\partial F}{\partial u}(x, 0) := (f_{ij}(x))_{n \times n}, \quad (1.3)$$

we assume that $f_{ij} \in C^2([0, 1])$

By the definition of hyperbolicity, we assume that $A(x, 0)$ is a diagonal matrix with distinct and nonzero eigenvalues $A(x, 0) = \text{diag}(\Lambda_1(x), \dots, \Lambda_n(x))$, which are, without loss of generality, ordered as follows:

$$\Lambda_1(x) < \Lambda_2(x) < \dots < \Lambda_m(x) < 0 < \Lambda_{m+1}(x) < \dots < \Lambda_n(x), \quad \forall x \in [0, 1]. \quad (1.4)$$

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Here and in what follows, $\text{diag}(\Lambda_1(x), \dots, \Lambda_n(x))$ denotes the diagonal matrix whose i -th element on the diagonal is $\Lambda_i(x)$.

Under the assumption (1.4), a general kind of boundary conditions which guarantee the well-posedness of the forward problem on the domain $\{(t, x) | t \geq 0, 0 \leq x \leq 1\}$ can be written as (see [15]):

$$x = 0 : u_s = G_s(u_1, \dots, u_m), \quad s = m + 1, \dots, n, \quad (1.5)$$

$$x = 1 : u_r = h_r(t), \quad r = 1, \dots, m, \quad (1.6)$$

where G_s are C^2 functions, and we assume that they vanish at the origin, i.e.

$$G_s(0, \dots, 0) \equiv 0, \quad s = m + 1, \dots, n, \quad (1.7)$$

while $H = (h_1, \dots, h_m)^T$ are boundary controls. Our concern, in this paper, is to design a feedback control law for $H(t)$ in order to ensure that the closed-loop system is locally exponentially stable in the H^2 norm.

In other words, we are interested in the following stabilization problem for the system (1.1) and (1.5)-(1.6):

Problem (ES). For any given $\lambda > 0$. Suppose that C^1 compatibility conditions are satisfied at the point $(t, x) = (0, 0)$. Does there exist a linear feedback control $\mathcal{B} : (H^2(0, 1))^n \rightarrow \mathbb{R}^m$, verifying the C^1 compatibility conditions at the point $(t, x) = (0, 1)$, such that for some $\varepsilon > 0$, the mixed initial-boundary value problem (1.1), (1.5)-(1.6) and the initial conditions

$$t = 0 : u(0, x) := \phi(x) = (\phi_1(x), \dots, \phi_n(x)), \quad (1.8)$$

with $H(t) = \mathcal{B}(u(t, \cdot))$ admits a unique $C^0([0, \infty); (H^2(0, 1))^n)$ solution $u = u(t, x)$, which satisfies

$$\|u(t, \cdot)\|_{H^2(0, L)} \leq C e^{-\lambda t} \|\phi(\cdot)\|_{H^2(0, L)}, \quad (1.9)$$

for some $C > 0$, provided that $\|\phi(\cdot)\|_{H^2(0, L)} \leq \varepsilon$?

The boundary stabilization problem for linear and nonlinear hyperbolic system has been widely studied in the last three decades or so. During this time, three parallel mathematical approaches have emerged. The first one is the so-called ‘‘Characteristic method’’, i.e. computing corresponding bounds by using explicit evolution of the solution along the characteristic curves. With this method, Problem (ES) has been previously investigated by Greenberg and Li (see [10]) for 2×2 systems and Li and Qin (see [15, 18]) for a generalization to $n \times n$ homogeneous systems in the framework of C^1 norm. Also, this method was developed by Li and Rao [16] to study the exact boundary controllability for general inhomogeneous quasilinear hyperbolic systems.

The second method is the ‘‘Control Lyapunov Functions method’’, which is a useful tool to analyze the asymptotic behavior of dynamical systems. This method was first used by Coron et.al. to design dissipative boundary conditions for nonlinear homogeneous hyperbolic systems in the context of both C^1 and H^2 norm [4, 5, 6]. More recently, it has been shown in [7] that the exponential stability strongly depends on the considered norm, i.e. a previously known sufficient condition for exponential stability with respect to the H^2 norm is not sufficient in the framework of C^1 norm. Although the Control Lyapunov Functions method has been introduced to study exponential stability for hyperbolic systems of balance laws, however, finding a ‘‘good’’ Lyapunov Function is the main difficulty, especially when the ‘‘natural’’ control Lyapunov functions do not lead to arbitrarily large exponential decay rate to the original system (see [1], [3, Pages 314 and 361–371]). This phenomenon indeed happens when we deal with Problem (ES) for the inhomogeneous hyperbolic systems (see [5] and [6]).

The third one is the ‘‘Backstepping method’’, which is now a popular mathematical tool to stabilize the finite dimensional and infinite dimensional dynamic systems (see [13, 14, 19, 20, 21]). In [8], a full-state feedback control law, with actuation on only one end of the domain, which achieves H^2 exponential stability of the closed-loop 2×2 linear and quasilinear hyperbolic

system is derived using a backstepping method. Moreover, this method ensures that the linear hyperbolic system vanishes in finite time. Unfortunately, the method presented in [8] can not be directly extended to $n \times n$ cases, especially when several states convecting in the same direction are controlled (see also [9]). In [11], a first step towards generalization to 3×3 linear hyperbolic systems is addressed, in the case where two controlled states are considered. With a similar Volterra transformation, designing an appropriate form of the target system, Hu et.al.[12] adopt a classical backstepping controller to handle the Problem (ES) for general $n \times n$ linear hyperbolic systems. Well-posedness of the system of kernel equations, which is the main technical challenge, is shown there by an improved successive approximation method.

In this paper, based on the results for the linear case [12], we will use the linearized feedback control to stabilize the nonlinear system as it is mentioned in [8]. Although the target system is a little different from the one in [8] with a linear term involved in the equations, thanks to its special structure, we show that all the procedures to handle nonlinearities in [8] can be also adapted in this paper with more technical developments. Let us recall some definitions and statements [8]. Define the norms

$$\begin{aligned}\|u(t, \cdot)\|_{H^1} &= \|u(t, \cdot)\|_{L^2} + \|u_x(t, \cdot)\|_{L^2}, \\ \|u(t, \cdot)\|_{H^2} &= \|u(t, \cdot)\|_{H^1} + \|u_{xx}(t, \cdot)\|_{L^2}.\end{aligned}$$

Our main result is given by

Theorem 1.1 *Under the assumptions in §1, suppose furthermore that C^1 compatibility conditions are satisfied at the point $(t, x) = (0, 0)$, there exists a continuous linear feedback control laws $\mathcal{B} : (H^2(0, 1))^n \rightarrow \mathbb{R}^m$, satisfying the C^1 compatibility conditions at the points $(t, x) = (0, 1)$, then for every $\lambda > 0$, there exist $\delta > 0$ and $c > 0$, such that the mixed initial-boundary value problem (1.1), (1.5), (1.6) and (1.8) with $H(t) = \mathcal{B}(u(t, \cdot))$ admits a unique $C^0([0, \infty), (H^2(0, 1))^n)$ solution $u = u(t, x)$, which verifies*

$$\|u(t, \cdot)\|_{H^2} \leq ce^{-\lambda t} \|\phi\|_{H^2}, \quad (1.10)$$

provided that $\|\phi\|_{H^2} \leq \delta$.

Remark 1.1 *The C^1 compatibility conditions at the point $(t, x) = (0, 0)$ are given by*

$$\phi_s(0) = G_s(\phi_1(0), \dots, \phi_m(0)) \quad s = m+1, \dots, n, \quad (1.11)$$

$$\begin{aligned}f_s(0, \phi(0)) - \sum_{j=1}^n a_{sj}(0, \phi(0))\phi'_j(0) = \\ \sum_{r=1}^m \frac{\partial G_s}{\partial u_r}(\phi_1(0), \dots, \phi_m(0)) \cdot \left(f_r(0, \phi(0)) - \sum_{j=1}^n a_{rj}(0, \phi(0))\phi'_j(0) \right) \quad s = m+1, \dots, n. \quad (1.12)\end{aligned}$$

The C^1 compatibility conditions at the point $(t, x) = (0, 1)$ are similar.

Remark 1.2 *For convenience, we always assume that the feedback controls $H(t) = \mathcal{B}(u(t, \cdot))$ satisfy the C^1 compatibility conditions at the point $(t, x) = (0, 1)$. However, if this property fails, one can add some dynamic terms to the controllers (see also Remark 3.1 and [8, Section 4]).*

The rest of this paper is organized as follows. In §2, we review a former result on the boundary backstepping controls for $n \times n$ linear hyperbolic system. Besides, we design a Lyapunov function to stabilize the linear system in the L^2 norm. In §3, we input the corresponding linearized closed-loop control to the original nonlinear system and give the feedback control design. In §4, we prove exponential stability of zero equilibrium for the quasilinear system by using the Control Lyapunov Function method. We finally include two appendices with some technical details.

2 Preliminaries–Linear Case

In this section, we review the results on stabilization of $n \times n$ hyperbolic linear system by using the backstepping method. Similar to the situation in [8], this procedure can be applied to locally stabilize the original nonlinear system. Consider the following $n \times n$ hyperbolic systems

$$w_t(t, x) + \Lambda(x)w_x(t, x) = \Sigma(x)w(t, x), \quad (2.1)$$

where, $w = (w_1, \dots, w_n)^T$ is a vector function of (t, x) , $\Lambda: [0, 1] \rightarrow \mathcal{M}_{n,n}(\mathbb{R})$ is an $n \times n$ C^2 diagonal matrix, i.e.

$$\Lambda(x) = \begin{pmatrix} \Lambda_-(x) & 0 \\ 0 & \Lambda_+(x) \end{pmatrix}, \quad (2.2)$$

in which $\Lambda_-(x) := \text{diag}(\lambda_1(x), \dots, \lambda_m(x))$ and $\Lambda_+(x) := \text{diag}(\lambda_{m+1}(x), \dots, \lambda_n(x))$ are diagonal submatrices, without loss of generality, satisfying

$$\lambda_1(x) < \dots < \lambda_m(x) < 0 < \lambda_{m+1}(x) < \dots < \lambda_n(x), \forall x \in [0, 1]. \quad (2.3)$$

On the other hand, $\Sigma: [0, 1] \rightarrow \mathcal{M}_{n,n}(\mathbb{R})$ is a $n \times n$ matrix with

$$\Sigma(x) = \begin{pmatrix} \Sigma^{--}(x) & \Sigma^{-+}(x) \\ \Sigma^{+-}(x) & \Sigma^{++}(x) \end{pmatrix}, \quad (2.4)$$

in which $\Sigma^{--} \in \mathcal{M}_{m,m}(\mathbb{R})$, $\Sigma^{-+} \in \mathcal{M}_{m,n-m}(\mathbb{R})$, $\Sigma^{+-} \in \mathcal{M}_{n-m,m}(\mathbb{R})$ and $\Sigma^{++} \in \mathcal{M}_{n-m,n-m}(\mathbb{R})$ are all C^2 submatrices with respect to x . Moreover, for any $i = 1, \dots, n$, we assume that

$$\Sigma_{ii}(x) \equiv 0, \quad \forall x \in [0, 1]. \quad (2.5)$$

The boundary conditions for the linear hyperbolic system (2.1) are given by

$$x = 0: w_+(t, 0) = Qw_-(t, 0), \quad (2.6)$$

and

$$x = 1: w_-(t, 1) = U(t). \quad (2.7)$$

where $w_- \in \mathbb{R}^m$, $w_+ \in \mathbb{R}^{n-m}$ are defined by requiring that $w := (w_-, w_+)^T$, $U = (U_1, \dots, U_m)^T$ are boundary feedback controls, $Q \in \mathcal{M}_{n-m,m}$ is a constant matrix. Our purpose in this section is to find a full-state feedback control law for $U(t)$ to ensure that the closed-loop system (2.1), (2.6)-(2.7) is globally asymptotically stable in the L^2 norm, which is defined by $\|w(t, \cdot)\|_{L^2} =$

$$\sqrt{\sum_{i=1}^n \int_0^1 w_i^2(t, x) dx}.$$

2.1 Target System

In Section 2.2, it will be shown that we can transform the system (2.1), (2.6)-(2.7) into the following cascade system

$$\gamma_t(t, x) + \Lambda(x)\gamma_x(t, x) = G(x)\gamma(t, 0) \quad (2.8)$$

with the boundary conditions

$$x = 0: \gamma_+(t, 0) = Q\gamma_-(t, 0) \quad (2.9)$$

and

$$x = 1 : \gamma_-(t, 1) = 0, \quad (2.10)$$

where $\gamma_- \in \mathbb{R}^m, \gamma_+ \in \mathbb{R}^{n-m}$ are defined by requiring that $\gamma := (\gamma_-, \gamma_+)^T$, G is a lower triangular matrix with following structure

$$G(x) = \begin{pmatrix} \mathcal{G}_1(x) & 0 \\ \mathcal{G}_2(x) & 0 \end{pmatrix}, \quad (2.11)$$

in which $\mathcal{G}_1 \in \mathcal{M}_{m,m}(\mathbb{R})$ is a lower triangular matrix, i.e.

$$\mathcal{G}_1(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}, \quad (2.12)$$

and $\mathcal{G}_2(x) \in \mathcal{M}_{n-m,m}(\mathbb{R})$. The coefficients of both \mathcal{G}_1 and \mathcal{G}_2 are to be determined in §2.2. Next, we prove that the cascade system (2.8)-(2.10) verifies the following proposition.

Proposition 2.1 *For any given matrix function $G(\cdot) \in C^1(0, 1)$, the mixed initial-boundary value problem (2.8)-(2.10) with initial condition*

$$t = 0 : \gamma(0, x) = \gamma_0(x), \quad (2.13)$$

where $\gamma_0 \in (L^2(0, 1))^n$ admits a $C^0([0, \infty); (L^2(0, 1))^n)$ solution $\gamma = \gamma(t, x)$, which is globally exponentially stable in the L^2 norm, i.e. for every $\lambda > 0$, there exists $c > 0$ such that

$$\|\gamma(t, \cdot)\|_{L^2} \leq ce^{-\lambda t} \|\gamma_0\|_{L^2}. \quad (2.14)$$

In fact, this solution vanishes in finite time $t > t_F$, where t_F is given by

$$t_F = \int_0^1 \frac{1}{\lambda_{m+1}(s)} + \sum_{r=1}^m \frac{1}{|\lambda_r(s)|} ds. \quad (2.15)$$

Proof. Equations (2.8) can be rewritten as

$$\begin{aligned} \partial_t \gamma_-(t, x) + \Lambda_-(x) \partial_x \gamma_-(t, x) &= \mathcal{G}_1(x) \gamma_-(t, 0), \\ \partial_t \gamma_+(t, x) + \Lambda_+(x) \partial_x \gamma_+(t, x) &= \mathcal{G}_2(x) \gamma_-(t, 0), \end{aligned} \quad (2.16)$$

then consider the following Lyapunov functional

$$V_0(t) = \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \gamma_+(t, x) dx - \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \gamma_-(t, x) dx, \quad (2.17)$$

where $\delta > 0$ is a parameter, $B = \text{diag}(b_1, \dots, b_m)$ (with $b_r > 0, r = 1, \dots, m$) whose coefficients are to be determined. Obviously, $\sqrt{V_0}$ is a norm equivalent to $\|\gamma(t, \cdot)\|_{L^2}$. Differentiating V_0 with respect to t and integrating by parts yields

$$\dot{V}_0(t) = I + II + III + IV$$

with

$$I = [-e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) + e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x)]_0^1,$$

$$\begin{aligned}
II &= - \int_0^1 \delta e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx - \int_0^1 \delta e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x) dx, \\
III &= 2 \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \mathcal{G}_2(x) \gamma_-(t, 0) dx, \\
IV &= -2 \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \mathcal{G}_1(x) \gamma_-(t, 0) dx.
\end{aligned}$$

Noting the boundary conditions (2.9)-(2.10), we have that

$$I = -e^\delta \gamma_+(t, 1)^T \gamma_+(t, 1) - \gamma_-(t, 0)^T (B - Q^T Q) \gamma_-(t, 0), \quad (2.18)$$

$$\begin{aligned}
III &\leq \int_0^1 e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx + \gamma_-(t, 0)^T \int_0^1 e^{-\delta x} \mathcal{G}_2^T(x) (\Lambda_+(x))^{-2} \mathcal{G}_2(x) dx \gamma_-(t, 0) \\
&\leq \int_0^1 e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx + \gamma_-(t, 0)^T \int_0^1 \mathcal{G}_2^T(x) (\Lambda_+(x))^{-2} \mathcal{G}_2(x) dx \gamma_-(t, 0),
\end{aligned} \quad (2.19)$$

$$\begin{aligned}
IV &= -2 \int_0^1 e^{\delta x} \sum_{m \geq i > j \geq 1} \gamma_i(t, x) \frac{b_i}{\Lambda_i(x)} g_{ij}(x) \gamma_j(t, 0) dx \\
&\leq -M \int_0^1 e^{\delta x} \sum_{m \geq i > j \geq 1} \frac{b_i}{\Lambda_i(x)} \gamma_i^2(t, x) dx - M \int_0^1 e^{\delta x} \sum_{m \geq i > j \geq 1} \frac{b_i}{\Lambda_i(x)} \gamma_j^2(t, 0) dx \\
&\leq -M \int_0^1 e^{\delta x} \sum_{m \geq i > j \geq 1} \frac{b_i}{\Lambda_i(x)} \gamma_i^2(t, x) dx + M \mu e^\delta \gamma_-(t, 0)^T \mathcal{C} \gamma_-(t, 0) \\
&\leq -mM \int_0^1 e^{\delta x} \sum_{i=2}^m \frac{b_i}{\Lambda_i(x)} \gamma_i^2(t, x) dx + M \mu e^\delta \gamma_-(t, 0)^T \mathcal{C} \gamma_-(t, 0) \\
&\leq -mM \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \gamma_-(t, x) dx + M \mu e^\delta \gamma_-(t, 0)^T \mathcal{C} \gamma_-(t, 0),
\end{aligned} \quad (2.20)$$

in which

$$M := \|G\|_{L^\infty}, \quad \mathcal{C} := \text{diag}(\mathcal{C}_1, \dots, \mathcal{C}_m) \quad (2.21)$$

with

$$\mathcal{C}_r := \begin{cases} \sum_{j=r+1}^m b_j, & 1 \leq r \leq m-1 \\ 0, & r = m, \end{cases} \quad (2.22)$$

and

$$\mu := \max_i \left\{ \frac{1}{\|\Lambda_i\|_{C^0}} \right\}. \quad (2.23)$$

Let

$$P = Q^T Q + \int_0^1 \mathcal{G}_2^T(x) (\Lambda_-(x))^{-2} \mathcal{G}_2(x) dx. \quad (2.24)$$

There exists a diagonal matrix $S = \text{diag}(s_1, \dots, s_m)$ with $s_r > 0$ ($r = 1, \dots, m$) being large enough, such that

$$P \prec S, \quad (2.25)$$

where $P \prec S$ denotes that $S - P$ is a positive-definite matrix. This yields

$$\begin{aligned}\dot{V}_0(t) &\leq -\gamma_-(t, 0)^T \left(B - S - M\mu e^{\delta \mathcal{C}} \right) \gamma_-(t, 0) - (\delta - 1) \int_0^1 e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx \\ &\quad - (\delta - mM\mu) \int_0^1 e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x) dx.\end{aligned}$$

Thus, for any given $\lambda > 0$, picking

$$\delta > \max \{ \lambda\mu + mM\mu, \lambda\mu + 1 \} \quad (2.26)$$

$$b_r > \begin{cases} M\mu e^{\delta} \sum_{j=r+1}^m b_j + s_r, & 1 \leq r \leq m-1 \\ s_m, & r = m, \end{cases} \quad (2.27)$$

we have

$$\dot{V}_0 \leq -\lambda V_0 \quad (2.28)$$

where λ can be chosen as large as desired. It is easy to see that Parameter matrix B does exist, since one can easily check (2.27) by induction. This shows exponential stability of γ system.

To show finite-time convergence to the origin, one can find the explicit solution of (2.8)-(2.10) as follows. Define

$$\phi_i(x) = \int_0^x \frac{1}{|\lambda_i(\xi)|} d\xi, \quad 1 \leq i \leq n. \quad (2.29)$$

Notice that every $\phi_i (1 \leq i \leq n)$ is monotonically increasing C^2 functions of x , and thus invertible. With the same statement in [8] and noting (2.8)-(2.12), one can express the explicit solution of γ_1 by

$$\gamma_1(t, x) = \begin{cases} \gamma_1(0, \phi_1^{-1}(\phi_1(x) + t)) & \text{if } t < \phi_1(1) - \phi_1(x), \\ 0 & \text{if } t \geq \phi_1(1) - \phi_1(x). \end{cases} \quad (2.30)$$

Notice in particular that γ_1 is identically zero for $t \geq \phi_1(1)$. From (2.8) and (2.12), we obtain that $\gamma_2(t, x)$ satisfies the following equation for $t \geq \phi_1(1)$

$$\partial_t \gamma_2(t, x) + \lambda_2(x) \partial_x \gamma_2(t, x) = 0, \quad (2.31)$$

with

$$\gamma_2(t, 1) = 0, \quad (2.32)$$

which ensures the explicit expression of $\gamma_2(t, x)$ to be

$$\gamma_2(t, x) = \begin{cases} \gamma_2(\phi_1(1), \phi_2^{-1}(\phi_2(x) + t)) & \text{if } \phi_1(1) < t < \phi_1(1) + \phi_2(1) - \phi_2(x), \\ 0 & \text{if } t \geq \phi_1(1) + \phi_2(1) - \phi_2(x). \end{cases} \quad (2.33)$$

Therefore, by induction, one has that $\gamma_r(t, x) (2 \leq r \leq m)$ satisfies the following equations, for $t > \sum_{k=1}^{r-1} \phi_k(1)$,

$$\partial_t \gamma_r(t, x) + \lambda_r(x) \partial_x \gamma_r(t, x) = 0, \quad (2.34)$$

with the boundary condition

$$\gamma_r(t, 1) = 0. \quad (2.35)$$

Thus, when $t > \sum_{k=1}^{r-1} \phi_k(1)$, we have

$$\gamma_r(t, x) = \begin{cases} \gamma_r(\sum_{k=1}^{r-1} \phi_k(1), \phi_r^{-1}(\phi_r(x) + t)) & \text{if } \sum_{k=1}^{r-1} \phi_k(1) < t < \sum_{k=1}^r \phi_k(1) - \phi_r(x), \\ 0 & \text{if } t \geq \sum_{k=1}^r \phi_k(1) - \phi_r(x). \end{cases} \quad (2.36)$$

This yields that $\gamma_-(t, x) \equiv 0$ ($t > \sum_{k=1}^m \phi_k(1)$). From the time $t = \sum_{k=1}^m \phi_k(1)$ on, we find γ_+ becomes the solution of the following system

$$\partial_t \gamma_+(t, x) + \Lambda_+(x) \partial_x \gamma_+(t, x) = 0 \quad (2.37)$$

with

$$x = 0 : \gamma_+(t, 0) \equiv 0. \quad (2.38)$$

Since (2.37)-(2.38) is a completely decoupled system, by the characteristic method, after $t = t_F$, where

$$t_F = \phi_{m+1}(1) + \sum_{r=1}^m \phi_r(1) = \int_0^1 \frac{1}{\lambda_{m+1}(s)} + \sum_{r=1}^m \frac{1}{|\lambda_r(s)|} ds, \quad (2.39)$$

one can see that $\gamma_+(t, x) \equiv 0$ ($t \geq t_F$), which concludes the Proof of Proposition 2.1. \blacksquare

2.2 Backstepping transformation and Kernel Equations

To map the original system (2.1) into the target system (2.8), we use the following Volterra transformation of the second kind, which is similar to the one in [8] and [9]:

$$\gamma(t, x) = w(t, x) - \int_0^x K(x, \xi) w(t, \xi) d\xi. \quad (2.40)$$

We point out here that this transformation yields that $w(t, 0) \equiv \gamma(t, 0)$ ($\forall t > 0$), which is crucial to design our feedback law.

Utilizing (2.1) and straightforward computations, one can show that

$$\begin{aligned} \gamma_t + \Lambda(x) \gamma_x = & - \int_0^x (K_\xi(x, \xi) \Lambda(\xi) + \Lambda(x) K_x(x, \xi) + K(x, \xi) \Sigma(\xi) + K(x, \xi) \Lambda_\xi(\xi)) w(t, \xi) d\xi \\ & + (\Sigma(x) + K(x, x) \Lambda(x) - \Lambda(x) K(x, x)) w(t, x) - K(x, 0) \Lambda(0) \begin{pmatrix} I & 0 \\ Q & 0 \end{pmatrix} w(t, 0). \end{aligned} \quad (2.41)$$

The original system (2.1) is mapped into the target system (2.8) if one has the following kernel equations:

$$\Lambda(x) K_x(x, \xi) + K_\xi(x, \xi) \Lambda(\xi) + K(x, \xi) \Sigma(\xi) + K(x, \xi) \Lambda_\xi(\xi) = 0 \quad (2.42)$$

$$\Sigma(x) + K(x, x) \Lambda(x) - \Lambda(x) K(x, x) = 0 \quad (2.43)$$

$$G(x) = -K(x, 0) \Lambda(0) \begin{pmatrix} I & 0 \\ Q & 0 \end{pmatrix} \quad (2.44)$$

Developing equations (2.42)–(2.44) leads to the following set of kernel PDEs

$$\lambda_i(x) \partial_x K_{ij}(x, \xi) + \lambda_j(\xi) \partial_\xi K_{ij}(x, \xi) = - \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj} \lambda_j'(\xi)) K_{ik}(x, \xi) \quad (2.45)$$

along with the following set of boundary conditions

$$K_{ij}(x, x) = \frac{\sigma_{ij}(x)}{\lambda_i(x) - \lambda_j(x)} \triangleq k_{ij}(x) \quad \text{for } 1 \leq i, j \leq n (i \neq j), \quad (2.46)$$

$$K_{ij}(x, 0) = -\frac{1}{\lambda_j(0)} \sum_{k=1}^{n-m} \lambda_{m+k}(0) K_{i, m+k}(x, 0) q_{k,j} \quad \text{for } 1 \leq i \leq j \leq m. \quad (2.47)$$

To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for $K_{ij}(m \geq i > j \geq 1, n \geq j > i \geq m+1)$ on $x = 1$:

$$K_{ij}(1, \xi) = k_{ij}^{(1)}(\xi), \quad \text{for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n, \quad (2.48)$$

and the boundary conditions for $K_{ij}(n \geq i \geq j \geq m+1)$ on $\xi = 0$:

$$K_{ij}(x, 0) = k_{ij}^{(2)}(x), \quad \text{for } m+1 \leq j \leq i \leq n. \quad (2.49)$$

where $k_{ij}^{(1)}$ and $k_{ij}^{(2)}$ are chosen as functions of $C^\infty[0, 1]$ satisfying the C^1 compatibility conditions at the point $(x, \xi) = (1, 1)$ (see Remark 2.1). The equations evolve in the triangular domain $\mathcal{T} = \{(x, \xi) : 0 \leq \xi \leq x \leq 1\}$. By Theorem A.1, one finds that there exists a unique piecewise $C^2(\mathcal{T})$ solution $K(x, \xi)$ to (2.45)-(2.49) with $K(x, x), K(x, 0) \in C^1(0, 1)$, provided that $\sigma_{ij}(x)$ are $C^2[0, 1]$, $\lambda_i(x)$ are $C^2[0, 1]$. While $G(x) \in C^1$ (with bounded C^1 norm) is given by (2.44) under the well-posedness of $K(x, 0)$, which is proved in Theorem A.1.

Remark 2.1 The C^1 compatibility conditions at the point $(x, \xi) = (1, 1)$ are given by

$$k_{ij}(1) = k_{ij}^{(1)}(1), \quad \text{for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n, \quad (2.50)$$

$$\dot{k}_{ij}^{(1)}(1) = \frac{\lambda_i(1)k_{ij}'(1) + \sum_{k=1}^n (\sigma_{kj}(1) + \delta_{kj}\lambda_j'(1))k_{ik}(1)}{\lambda_i(1) - \lambda_j(1)}, \quad \text{for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n. \quad (2.51)$$

2.3 The inverse transformation and stabilization for linear system

Transformation (2.40) is a classical Volterra equation of the second kind, one can check from Theorem A.2 that there exists a unique piecewise $C^2(\mathcal{T})$ matrix function $L(x, \xi)$ such that

$$w(t, x) = \gamma(t, x) + \int_0^x L(x, \xi) \gamma(t, \xi) d\xi. \quad (2.52)$$

From the transformation (2.40) evaluated at $x = 1$, one gets the following feedback control laws

$$U_i(t) = \int_0^1 \sum_{j=1}^n K_{ij}(1, \xi) w_j(t, \xi) d\xi, \quad (i = 1, \dots, m), \quad (2.53)$$

which immediately leads to our feedback stabilization result for the linear system as follows:

Theorem 2.1 The mixed initial-boundary value problem (2.1) with the boundary conditions (2.6), the feedback control law (2.53) and initial condition

$$t = 0 : w(0, x) = w_0(x), \quad (2.54)$$

in which $w_0 \in (L^2(0, 1))^n$, admits a $(L^2(0, 1))^n$ solution $w = w(t, x)$. Moreover, for every $\eta > 0$, there exists $c > 0$ such that

$$\|w(\cdot, t)\|_{L^2} \leq ce^{-\eta t} \|w_0\|_{L^2}. \quad (2.55)$$

In fact, w vanishes in finite time $t > t_F$, where t_F is given by (2.15).

Remark 2.2 If we focus on the linear problem, Λ and Σ can be assumed to be $C^1([0, 1])$ and $C^0([0, 1])$ functions. The corresponding kernels K and L are then both functions of $L^\infty(\mathcal{T})$.

3 Backstepping boundary control design for nonlinear system

As mentioned in [8], we wish the linear controller (2.53) designed by backstepping method to work locally for the corresponding nonlinear system. Let us show that this is indeed the case. Introduce

$$\varphi_i(x) := \exp \left(- \int_0^x \frac{f_{ii}(s)}{\Lambda_i(s)} ds \right) \quad i = 1, \dots, n. \quad (3.1)$$

One can make the following coordinates transformation

$$w(t, x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} u(t, x) = \Phi(x) u(t, x). \quad (3.2)$$

Then the original control system u is transformed into the following system expressed in the new coordinates:

$$w_t(t, x) + \bar{A}(x, w) w_x(t, x) = \tilde{F}(x, w), \quad (3.3)$$

in which

$$\bar{A}(x, w) = \Phi(x) A(x, \Phi^{-1}(x) w) \Phi^{-1}(x), \quad (3.4)$$

$$\tilde{F}(x, w) = \Phi(x) F(x, \Phi^{-1}(x) w) - \bar{A}(x, w) \begin{pmatrix} \frac{f_{11}(x)}{\Lambda_1(x)} \\ \vdots \\ \frac{f_{nn}(x)}{\Lambda_n(x)} \end{pmatrix} w. \quad (3.5)$$

Obviously, one can check that

$$\tilde{F}(x, 0) = 0, \quad (3.6)$$

$$\bar{A}(x, 0) = \Phi(x) A(x, 0) \Phi^{-1}(x) = A(x, 0). \quad (3.7)$$

Moreover, define

$$\Sigma(x) = \left. \frac{\partial \tilde{F}(x, w)}{\partial w} \right|_{w=0}, \quad (3.8)$$

we have that

$$\Sigma_{ij}(x) = \begin{cases} \frac{\varphi_i(x)}{\varphi_j(x)} f_{ij}(x), & i \neq j, \\ 0, & i = j. \end{cases} \quad (3.9)$$

Therefore, we may rewrite (3.3) as a linear system with the same structure as (2.1) plus nonlinear terms:

$$w_t(t, x) + \Lambda(x) w_x(t, x) = \Sigma(x) w(t, x) + \Lambda_{NL}(x, w) w_x(t, x) + f_{NL}(x, w), \quad (3.10)$$

where

$$\Lambda(x) = A(x, 0), \quad (3.11)$$

and

$$\Lambda_{NL}(x, w) = \Lambda(x) - \bar{A}(x, w), \quad f_{NL}(x, w) = \tilde{F}(x, w) - \Sigma(x) w(t, x). \quad (3.12)$$

For the boundary conditions of the system (3.10), defining

$$Q = \left(\frac{\partial G_s}{\partial u_r} \right)_{(n-m) \times m} \Big|_{u=0} \quad \text{and} \quad G_{NL}(w_-(t, 0)) = G(w_-(t, 0)) - Qw_-(t, 0), \quad (3.13)$$

one obtains that

$$x = 0 : w_+(t, 0) = Qw_-(t, 0) + G_{NL}(w_-(t, 0)) \quad (3.14)$$

and

$$x = 1 : w_-(t, 1) = U(t), \quad (3.15)$$

where

$$U(t) = \begin{pmatrix} \varphi_1(1) & & \\ & \ddots & \\ & & \varphi_m(1) \end{pmatrix} H(t) = \tilde{\Phi}(1)H(t). \quad (3.16)$$

It is easily verified that

$$\Lambda(x, 0) = 0, \quad f_{NL}(x, 0) = \frac{\partial f_{NL}}{\partial w}(x, 0) = 0 \quad (3.17)$$

and

$$G_{NL}(0) = \frac{\partial G_{NL}}{\partial w}(0) = 0. \quad (3.18)$$

Thus, the feedback control law can be chosen as

$$h_r(t) = \tilde{\Phi}_{rr}^{-1}(1)U_r(t) = \tilde{\Phi}_{rr}^{-1}(1) \int_0^1 \sum_{j=1}^n K_{rj}(1, \xi) \tilde{\Phi}_{jj}(\xi) u_j(t, \xi) d\xi, \quad r = 1, \dots, m, \quad (3.19)$$

where the kernels are computed from (2.45)–(2.49) with the coefficients $\Sigma(x)$ and $\Lambda(x)$ obtained from (3.9) and (3.11). One easily verifies that under the assumptions of §1, both Σ and Λ are functions of C^2 .

Remark 3.1 *The C^1 compatibility conditions at the point $(t, x) = (0, 1)$ for system (1.1) with boundary conditions (3.15) should be*

$$\phi_r(1) = \sum_{j=1}^n \int_0^1 \tilde{k}_{rj}(\xi) \phi_j(\xi) d\xi, \quad r = 1, \dots, m, \quad (3.20)$$

$$\begin{aligned} f_r(1, \phi(1)) - \sum_{j=1}^n a_{rj}(1, \phi(1)) \phi_j'(1) = \\ \sum_{k=1}^n \int_0^1 \tilde{k}_{rk}(\xi) \left(f_k(1, \phi(1)) - \sum_{j=1}^n a_{kj}(1, \phi(1)) \phi_j'(1) \right), \quad r = 1, \dots, m, \end{aligned} \quad (3.21)$$

where $\tilde{k}_{rk}(\xi)$ are the elements of the matrix $\tilde{K}(\xi)$ with

$$\tilde{K}(\xi) = \tilde{\Phi}^{-1}(1)K(1, \xi)\tilde{\Phi}(\xi). \quad (3.22)$$

Notice that (3.20)-(3.21) depend on the feedback control design, however, there are no physical reasons that the initial data should satisfy them. In order to guarantee the initial conditions independent of these artificial conditions, we, following [8], modify the boundary controls on $x = 1$ as

$$x = 1: \quad u_r = h_r(t) + a_r(t) + b_r(t), \quad r = 1, \dots, m, \quad (3.23)$$

where a_r and b_r are the state of the following dynamic systems

$$\dot{a}_r(t) = -d_r a_r(t), \quad \dot{b}_r(t) = -\tilde{d}_r b_r(t), \quad r = 1, \dots, m \quad (3.24)$$

with $d_r > 0, \tilde{d}_r > 0$ and $d_r \neq \tilde{d}_r, r = 1, \dots, m$. By the modified control designs (3.23), the compatibility conditions on $x = 1$ are rewritten by

$$\phi_r(1) = \sum_{j=1}^n \int_0^1 \tilde{k}_{rj}(\xi) \phi_j(\xi) d\xi + a_r(0) + b_r(0), \quad r = 1, \dots, m, \quad (3.25)$$

$$\begin{aligned} f_r(1, \phi(1)) - \sum_{j=1}^n a_{rj}(1, \phi(1)) \phi_j'(1) = \\ \sum_{k=1}^n \int_0^1 \tilde{k}_{rk}(\xi) \left(f_k(1, \phi(1)) - \sum_{j=1}^n a_{kj}(1, \phi(1)) \phi_j'(1) \right) - d_r a_r(0) - \tilde{d}_r b_r(0), \quad r = 1, \dots, m. \end{aligned} \quad (3.26)$$

For any $1 \leq r \leq m$, call

$$\mathcal{P}_r(\phi) = \phi_r(1) - \sum_{j=1}^n \int_0^1 \tilde{k}_{rj}(\xi) \phi_j(\xi) d\xi \quad (3.27)$$

$$\mathcal{M}_r(\phi) = f_r(1, \phi(1)) - \sum_{j=1}^n a_{rj}(1, \phi(1)) \phi_j'(1) - \sum_{k=1}^n \int_0^1 \tilde{k}_{rk}(\xi) \left(f_k(1, \phi(1)) - \sum_{j=1}^n a_{kj}(1, \phi(1)) \phi_j'(1) \right) \quad (3.28)$$

Picking

$$a_r(0) = -\frac{\mathcal{M}_r(\phi) + \tilde{d}_r \mathcal{P}_r(\phi)}{d_r - \tilde{d}_r}, \quad b_r(0) = \frac{d_r \mathcal{P}_r(\phi) + \mathcal{M}_r(\phi)}{d_r - \tilde{d}_r}, \quad (3.29)$$

the compatibility conditions are automatically verified. Similar stabilization results as Theorem 1.1 are still valid for the closed-loop system (1.1), (1.5) and (3.23) (see [8, Theorem 4.1]). In fact, this dynamic extension is designed to avoid restriction for artificial boundary conditions due to the compatibility conditions at the points $(t, x) = (0, 1)$, and it has been introduced in [2] to deal with the stabilization of the Euler equations of incompressible fluids (see also [22]).

4 Proof of Theorem 1.1

In this section, we will prove the exponential stability for the system (1.1), (1.5) and (1.6) under the boundary feedback controls (3.19) by Control Lyapunov Function method. The whole proof is divided into the following steps.

4.1 Definitions

We first define some notations (omitting the time argument):

$$\|\gamma\|_\infty := \operatorname{ess\,sup}_{x \in [0,1]} |\gamma(x)|, \quad \|\gamma\|_{L^p} := \left(\int_0^1 |\gamma(\xi)|^p d\xi \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty. \quad (4.1)$$

For a $n \times n$ matrix, denote

$$|M| := \max\{\|M\gamma\|_{L^\infty} : \gamma \in \mathbb{R}^n, |\gamma| = 1\}. \quad (4.2)$$

For a piecewise kernel matrix $K(x, \xi)$, which is a continuous function on each domain D_i ($i = 1, \dots, \mathcal{S}$), respectively, with

$$\mathcal{T} = \bigcup_{i=1}^{\mathcal{S}} D_i, \quad (4.3)$$

$$D_i \cap D_j = \emptyset, (i \neq j). \quad (4.4)$$

Let

$$\|K\|_\infty := \max_i \sup_{(x, \xi) \in D_i} |K(x, \xi)|. \quad (4.5)$$

As before, we recall the following symbols of [8] for simplicity:

$$\mathcal{K}[\gamma](t, x) = \gamma(t, x) - \int_0^x K(x, \xi) \gamma(t, \xi) d\xi, \quad (4.6)$$

$$\mathcal{L}[\gamma](t, x) = \gamma(t, x) + \int_0^x L(x, \xi) \gamma(t, \xi) d\xi, \quad (4.7)$$

$$\mathcal{K}_1[\gamma](t, x) = -K(x, x) \gamma(t, x) + \int_0^x K_\xi(x, \xi) \gamma(t, \xi) d\xi, \quad (4.8)$$

$$\mathcal{K}_2[\gamma](t, x) = -K(x, x) \gamma(t, x) - \int_0^x K_x(x, \xi) \gamma(t, \xi) d\xi, \quad (4.9)$$

$$\mathcal{L}_1[\gamma](t, x) = L(x, x) \gamma(t, x) + \int_0^x L_x(x, \xi) \gamma(t, \xi) d\xi. \quad (4.10)$$

Define $F_1[\gamma]$ and $F_2[\gamma]$ as

$$F_1[\gamma] := \Lambda_{NL}(x, \mathcal{L}[\gamma]), \quad F_2[\gamma] := f_{NL}(x, \mathcal{L}[\gamma]). \quad (4.11)$$

To prove our result, we notice that if we apply the (inverse) backstepping transformation (2.40) to the nonlinear system (3.10), we obtain the following transformed system

$$\begin{aligned} & \gamma_t(t, x) + \Lambda(x) \gamma_x(t, x) - G(x) \gamma(t, 0) \\ &= \mathcal{K}[\Lambda_{NL}(x, w) w_x] + \mathcal{K}[f_{NL}(x, w)] \\ &= \mathcal{K}[\Lambda_{NL}(x, w) \gamma_x] + \mathcal{K}[\Lambda_{NL}(x, w) \mathcal{L}_1[\gamma]] + \mathcal{K}[f_{NL}(x, w)] \\ &= F_3[\gamma, \gamma_x] + F_4[\gamma], \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} F_3 &= \mathcal{K}[F_1[\gamma] \gamma_x], \\ F_4 &= \mathcal{K}[F_1[\gamma] \mathcal{L}_1[\gamma] + F_2[\gamma]]. \end{aligned}$$

The boundary conditions are

$$x = 0 : \gamma_+(t, 0) = Q \gamma_-(t, 0) + G_{NL}(\gamma_-(t, 0)) \quad (4.13)$$

and

$$x = 1 : \gamma_-(t, 1) = 0. \quad (4.14)$$

Notice that here we may lose the regularity on the point $(0, 0)$ for the kernels K and L , which leads both of them to be discontinuous (see [12]). However, by the assumptions on the coefficients and applying Theorem A.1 and Theorem A.2, the direct and inverse transformations (2.40) and (2.52) have C^2 piecewise kernels functions. Fortunately, differentiating twice with respect to x in these transformations, by the similar argument in [8] and [22, Proposition 3.1] as well as the additive property of the integral, it can be shown that the H^2 norm of γ is equivalent to the H^2 norm of w . Thus, if we show H^2 local stability of the origin for (4.12)-(4.14), the same holds for w i.e. u .

In order to get the desired H^2 estimation for γ , the things left are just estimating the growth of $\|\gamma\|_{L^2}$, $\|\gamma_t\|_{L^2}$ and $\|\gamma_{tt}\|_{L^2}$, respectively.

4.2 Analyzing the growth of $\|\gamma\|_{L^2}$

Let

$$F_3[\gamma, \gamma_x] = (F_3^-[\gamma, \gamma_x], F_3^+[\gamma, \gamma_x])^T, \quad F_4[\gamma] = (F_4^-[\gamma], F_4^+[\gamma])^T. \quad (4.15)$$

where F_3^- and $F_4^- \in \mathbb{R}^m$, F_3^+ and $F_4^+ \in \mathbb{R}^{n-m}$.

Define

$$V_1(t) = \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \gamma_+(t, x) dx - \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \gamma_-(t, x) dx. \quad (4.16)$$

Differentiating V_1 with respect to time and integrating by parts yields

$$\dot{V}_1(t) = V + VI + VII + VIII + IX + X$$

with

$$\begin{aligned} V &= [-e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) + e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x)]_0^1, \\ VI &= - \int_0^1 \delta e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx - \int_0^1 \delta e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x) dx, \\ VII &= 2 \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \mathcal{G}_2(x) \gamma_-(t, 0) dx, \\ VIII &= -2 \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \mathcal{G}_1(x) \gamma_-(t, 0) dx, \\ IX &= 2 \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} (F_3^+[\gamma, \gamma_x] + F_4^+[\gamma]) dx, \\ X &= -2 \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} (F_3^-[\gamma, \gamma_x] + F_4^-[\gamma]) dx. \end{aligned}$$

By the same argument in [8] and noting Lemma B.2, we have

$$\begin{aligned} IX + X &\leq K_1 \int_0^1 |\gamma| (|F_3[\gamma, \gamma_x]| + |F_4[\gamma]|) dx \\ &\leq K_2 (\|\gamma_x\|_\infty V_1 + V_1^{\frac{3}{2}}). \end{aligned} \quad (4.17)$$

Moreover, for $\|\gamma\|_\infty \leq \delta$, $|G_{NL}(\gamma_-(t, 0))| \leq K_3 |\gamma_-(t, 0)|$, then

$$\begin{aligned} V &= -e^{-\delta} \gamma_+(t, 1)^T \gamma_+(t, 1) + e^{\delta} \gamma_-(t, 1)^T B \gamma_-(t, 1) + \gamma_+(t, 0)^T \gamma_+(t, 0) - \gamma_-(t, 0)^T B \gamma_-(t, 0) \\ &\leq -\gamma_-(t, 0)^T (B - Q^T Q - K_3^2 I_m) \gamma_-(t, 0). \end{aligned} \quad (4.18)$$

By (2.19) and (2.20), one immediately obtains

$$\begin{aligned}\dot{V}_1(t) \leq & -\gamma_-(t,0)^T \left(B - \tilde{S} - M\mu e^{\delta} \mathcal{C} \right) \gamma_-(t,0) - (\delta - 1) \int_0^1 e^{-\delta x} \gamma_+(t,x)^T \gamma_+(t,x) dx \\ & - (\delta - mM\mu) \int_0^1 e^{\delta x} \gamma_-(t,x)^T B \gamma_-(t,x) dx + K_2 (V_1^{\frac{3}{2}} + \|\gamma_x\|_{\infty} V_1),\end{aligned}$$

where M, \mathcal{C}, μ are given by (2.21) and (2.23), $\tilde{S} := S + K_3^2 I_m$ with S stated in (2.25). Thus, for any given $\lambda_1 > 0$, picking

$$\delta > \max \{ \lambda_1 \mu + mM\mu, \lambda_1 \mu + 1 \}, \quad (4.19)$$

$$b_r := \begin{cases} M\mu e^{\delta} \sum_{j=r+1}^m b_j + \tilde{s}_r, & 1 \leq r \leq m-1 \\ \tilde{s}_m, & r = m, \end{cases} \quad (4.20)$$

we have the following

Proposition 4.1 *For any given $\lambda_1 > 0$, there exists $\delta_1 > 0$ and $K_2 > 0$, such that*

$$\dot{V}_1 \leq -\lambda_1 V_1 + K_2 (V_1^{\frac{3}{2}} + \|\gamma_x\|_{\infty} V_1), \quad (4.21)$$

provided $\|\gamma\|_{\infty} \leq \delta_1$.

4.3 Analyzing the growth of $\|\gamma_t\|_{L^2}$

Let $\zeta = \gamma_t$. Taking the partial derivative with t in (4.12) yields:

$$\zeta_t(t,x) + (\Lambda(x) - F_1[\gamma])\zeta_x(t,x) - G(x)\zeta(t,0) = F_5[\gamma, \gamma_x, \zeta] + F_6[\gamma, \zeta], \quad (4.22)$$

where

$$F_5 = \mathcal{K}_1[F_1[\gamma]\zeta] + \int_0^x K(x,\xi) F_{12}[\gamma, \gamma_x]\zeta(\xi) d\xi + K(x,0) \Lambda_{NL}(0, \gamma(0))\zeta(0) + \mathcal{K}[F_{11}[\gamma, \zeta]\gamma_x], \quad (4.23)$$

$$F_6 = \mathcal{K}[F_{11}[\gamma, \zeta]\mathcal{L}_1[\gamma]] + \mathcal{K}[F_1[\gamma]\mathcal{L}_1[\zeta]] + \mathcal{K}[F_{21}[\gamma, \zeta]], \quad (4.24)$$

with

$$\begin{aligned}F_{11} &= \frac{\partial \Lambda_{NL}}{\partial \gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta], \\ F_{12} &= \frac{\partial \Lambda_{NL}}{\partial \gamma}(x, \mathcal{L}[\gamma])(\gamma_x + \mathcal{L}_1[\gamma]) + \frac{\partial \Lambda_{NL}}{\partial \gamma}(x, \mathcal{L}[\gamma]), \\ F_{21} &= \frac{\partial f_{NL}}{\partial \gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta].\end{aligned} \quad (4.25)$$

The boundary conditions are given by

$$x = 0 : \zeta_+(t,0) = Q\zeta_-(t,0) + \frac{\partial G_{NL}}{\partial \gamma_-}(\gamma_-(t,0))\zeta_-(t,0) \quad (4.26)$$

and

$$x = 1 : \zeta_-(t,1) = 0, \quad (4.27)$$

in which $\zeta_- \in \mathbb{R}^m, \zeta_+ \in \mathbb{R}^{n-m}$ are defined by requiring that $\zeta := (\zeta_-, \zeta_+)^T$.

Similarly as in [8], we need the following lemma in order to find a Lyapunov function for $\zeta(t,x)$:

Lemma 4.1 *There exists $\delta > 0$ such that, for any $\|\gamma\|_\infty \leq \delta$, there exists a symmetric matrix $R[\gamma]$ satisfying the identity*

$$R[\gamma](\Lambda(x) - F_1[\gamma]) - (\Lambda(x) - F_1[\gamma])^T R[\gamma] = 0. \quad (4.28)$$

Moreover, we have that

$$|R[\gamma](x)| \leq c_1 + c_2 \|\gamma\|_\infty, \quad (4.29)$$

$$\left| ((R[\gamma] - D(x))\Lambda(x))_x \right| \leq c_2(\|\gamma\|_\infty + \|\gamma_x\|_\infty), \quad (4.30)$$

$$|(R[\gamma])_t| \leq c_3(|\zeta| + \|\zeta\|_{L^1}), \quad (4.31)$$

where c_1 , c_2 and c_3 are positive constants, and

$$D(x) = \begin{pmatrix} -e^{\delta x} B(\Lambda_-(x))^{-1} & 0 \\ 0 & e^{-\delta x} (\Lambda_+(x))^{-1} \end{pmatrix}. \quad (4.32)$$

Proof: Denote $\mathcal{D}_n(x)$ as the set of $n \times n$ diagonal matrices with C^1 elements. Let $\Lambda(x) := \text{diag}(\Lambda_1(x), \dots, \Lambda_n(x)) \in \mathcal{D}_n(x)$ be such that $\Lambda_i(x) \neq \Lambda_j(x) (i \neq j \forall x \in [0, 1])$ holds. Notice that $D \in \mathcal{D}_n(x)$. Based on the proof in [5, Lemma 4.1], one can easily see that there exist a positive real number η and a map $\mathcal{N} : \{M \in \mathcal{M}_{n,n}(\mathbb{R}; x); \|M(x) - \Lambda(x)\|_{C^1} < \eta\} \rightarrow \mathcal{S}_n$ of class C^∞ such that

$$\mathcal{N}(\Lambda(x)) = D(x), \quad (4.33)$$

and

$$\mathcal{N}(M)M - M^T \mathcal{N}(M) = 0 \quad \forall M \in \mathcal{M}_{n,n}(\mathbb{R}; x), \quad \|M(x) - \Lambda(x)\|_{C^1} < \eta. \quad (4.34)$$

It then suffices to define $R[\gamma]$ by

$$R[\gamma] = \mathcal{N}(\Lambda(x) - F_1[\gamma]). \quad (4.35)$$

Moreover, by the regularity of \mathcal{N} and Lemma B.2–B.3, one can show that

$$\begin{aligned} |R[\gamma]| &\leq |D(x)| + |R[\gamma] - D(x)| \\ &\leq c_4 + c_5 |F_1[\gamma]| \\ &\leq c_4 + c_6 \|\gamma\|_\infty, \end{aligned} \quad (4.36)$$

$$\begin{aligned} \left| ((R[\gamma] - D(x))\Lambda(x))_x \right| &\leq |(R[\gamma] - D(x))_x \Lambda(x)| + |(R[\gamma] - D(x))\Lambda_x(x)| \\ &\leq c_7 |F_{12}| + c_8 |F_1| \\ &\leq c_9(\|\gamma\|_\infty + \|\gamma_x\|_\infty) \end{aligned} \quad (4.37)$$

and

$$|R[\gamma]_t| \leq c_{10} \left| \frac{\partial F_1[\gamma]}{\partial t} \right| \quad (4.38)$$

$$\leq c_{10} |F_{11}[\gamma, \zeta]| \quad (4.39)$$

$$\leq c_{11}(|\zeta| + \|\zeta\|_{L^1}). \quad (4.40)$$

This concludes the proof of Lemma 4.1. ■

Define

$$V_2(t) = \int_0^1 \zeta^T(t, x) R[\gamma] \zeta(t, x) dx. \quad (4.41)$$

Using (4.28) and straightforward computations, one can show that

$$\dot{V}_2(t) = XI + XII + XIII + XIV + XV$$

with

$$\begin{aligned} XI &= \int_0^1 \zeta^T(t, x) (R[\gamma] (\Lambda(x) - F_1[\gamma]))_x \zeta(t, x) dx, \\ XII &= - [\zeta^T(t, x) R[\gamma] (\Lambda(x) - F_1[\gamma]) \zeta(t, x)]_{x=0}^{x=1}, \\ XIII &= \int_0^1 \zeta(t, x) (R[\gamma])_t \zeta(t, x) dx, \\ XIV &= 2 \int_0^1 \zeta^T(t, x) R[\gamma] F_5[\gamma, \gamma_x, \zeta, \zeta_x] dx + 2 \int_0^1 \zeta^T(t, x) R[\gamma] F_6[\gamma, \zeta] dx, \\ XV &= 2 \int_0^1 \zeta^T(t, x) R[\gamma] G(x) \zeta(t, 0) dx. \end{aligned}$$

For XII and XV , by the boundary conditions (4.26)–(4.27), we have

$$\begin{aligned} XII + XV &= - [\zeta^T(t, x) (D(x) + \Theta[\gamma]) (\Lambda(x) - F_1[\gamma]) \zeta(t, x)]_{x=0}^{x=1} \\ &\quad + 2 \int_0^1 \zeta^T(t, x) (D(x) + \Theta[\gamma]) G(x) \zeta(t, 0) dx \\ &= - [\zeta^T(t, x) (D(x) \Lambda(x) + \Theta[\gamma] \Lambda(x) - D(x) F_1[\gamma] - \Theta[\gamma] F_1[\gamma]) \zeta(t, x)]_{x=0}^{x=1} \\ &\quad + 2 \int_0^1 \zeta^T(t, x) D(x) G(x) \zeta(t, 0) dx + 2 \int_0^1 \zeta^T(t, x) \Theta[\gamma] G(x) \zeta(t, 0) dx \\ &\leq -\zeta_-(t, 0)^T (B - \tilde{S} - M\mu e^\delta \mathcal{C} - K_3 \|\gamma\|_\infty I_m) \zeta_-(t, 0) \\ &\quad + \int_0^1 e^{-\delta x} \zeta_+(t, x)^T \zeta_+(t, x) dx + mM\mu \int_0^1 e^{\delta x} \zeta_-(t, x)^T B \zeta_-(t, x) dx \\ &\quad + K_4 \|\gamma\|_\infty V_2. \end{aligned} \quad (4.42)$$

As stated in [8], we obtain

$$XI \leq -\lambda_2 V_2 + K_4 \|\zeta\|_{L^2}^2 (\|\gamma\|_\infty + \|\gamma_x\|_\infty), \quad (4.43)$$

$$XIII \leq K_5 \|\zeta\|_{L^2}^2 \|\zeta\|_\infty, \quad (4.44)$$

$$XIV \leq K_6 \left(\|\zeta\|_{L^2}^2 (\|\gamma\|_\infty + \|\gamma_x\|_\infty) + \|\zeta\|_{L^2} |\zeta(t, 0)| |\gamma(t, 0)| \right). \quad (4.45)$$

Following Lemma B.5, we are in the position to conclude that

Proposition 4.2 *For any given $\lambda_2 > 0$, there exists $\delta_2 > 0$ and $K_7 > 0$, such that*

$$\dot{V}_2 \leq -\lambda_2 V_2 + K_7 (\|\zeta\|_\infty + \|\gamma\|_\infty) V_2, \quad (4.46)$$

provided that $\|\gamma\|_\infty \leq \delta_2$.

4.4 Analyzing the growth of $\|\gamma_{tt}\|_{L^2}$

We next deal with $\|\gamma_{tt}\|_{L^2}$. Define $\theta = \gamma_{tt}$. Taking a partial derivative with respect to t for (4.22), one obtains an equation of θ :

$$\theta_t + [\Lambda(x) - F_1[\gamma]]\theta_x = G(x)\theta(t, 0) + F_7[\gamma, \gamma_x, \zeta, \zeta_x, \theta] + F_8[\gamma, \zeta, \theta], \quad (4.47)$$

where

$$\begin{aligned} F_7 = & \mathcal{K}_1[F_{11}[\gamma, \zeta]\zeta] + \int_0^x K(x, \xi)F_{12}[\gamma, \gamma_x]\theta(\xi)d\xi + \mathcal{K}_1[F_1[\gamma]\theta] \\ & + \int_0^x K(x, \xi)F_{14}[\gamma, \gamma_x, \zeta, \zeta_x]\zeta(\xi)d\xi + K(x, 0)\frac{\partial\Lambda_{NL}}{\partial\gamma}(0, \gamma(0))\zeta(0)\zeta(0) \\ & + K(x, 0)\Lambda_{NL}(0, \gamma(0))\theta(0) + \mathcal{K}[F_{11}[\gamma, \zeta]\zeta_x] + \mathcal{K}[F_{13}[\gamma, \zeta, \theta]\gamma_x], \end{aligned} \quad (4.48)$$

$$F_8 = 2\mathcal{K}[F_{11}[\gamma, \zeta]\mathcal{L}_1[\zeta]] + \mathcal{K}[F_1[\gamma]\mathcal{L}_1[\theta]] + \mathcal{K}[F_{13}[\gamma, \zeta, \theta]\mathcal{L}_1[\gamma]] + \mathcal{K}[F_{22}[\gamma, \zeta, \theta]] \quad (4.49)$$

with

$$F_{13} = \frac{\partial\Lambda_{NL}^2}{\partial\gamma^2}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta]\mathcal{L}[\zeta] + \frac{\partial\Lambda_{NL}}{\partial\gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\theta], \quad (4.50)$$

$$F_{14} = \frac{\partial\Lambda_{NL}^2}{\partial\gamma^2}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta](\gamma_x + \mathcal{L}_1[\gamma]) + \frac{\partial\Lambda_{NL}}{\partial\gamma}(x, \mathcal{L}[\gamma])(\zeta_x + \mathcal{L}_1[\zeta]) + \frac{\partial^2\Lambda_{NL}}{\partial x\partial\gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta], \quad (4.51)$$

$$F_{22} = \frac{\partial^2 f_{NL}}{\partial\gamma^2}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta]\mathcal{L}[\zeta] + \frac{\partial f_{NL}}{\partial\gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\theta]. \quad (4.52)$$

The boundary conditions of θ are given by

$$x = 0 : \theta_+(t, 0) = Q\theta_-(t, 0) + \frac{\partial G_{NL}}{\partial\gamma_-}(\gamma_-(t, 0))\theta_-(t, 0) + \frac{\partial^2 G_{NL}}{\partial\gamma_-^2}(\gamma_-(t, 0))\zeta_-(t, 0)\zeta_-(t, 0) \quad (4.53)$$

and

$$x = 1 : \theta_-(t, 1) = 0. \quad (4.54)$$

where $\theta_- \in \mathbb{R}^m, \theta_+ \in \mathbb{R}^{n-m}$ are defined by requiring that $\theta := (\theta_-, \theta_+)^T$.

In order to control $\|\theta\|_{L^2}$, we introduce

$$V_3(t) = \int_0^1 \theta^T(t, x)R[\gamma]\theta(t, x)dx, \quad (4.55)$$

then it is easy to see that

$$\dot{V}_3(t) = XVI + XVII + XVIII + XIX + XX \quad (4.56)$$

with

$$\begin{aligned} XVI = & \int_0^1 \theta^T(t, x)(R[\gamma](\Lambda(x) - F_1[\gamma]))_x\theta(t, x)dx, \\ XVII = & -[\theta^T(t, x)R[\gamma](x)(\Lambda(x) - F_1[\gamma](x))\theta(t, x)]_{x=0}^{x=1}, \\ XVIII = & \int_0^1 \theta^T(t, x)(R[\gamma])_t\theta(t, x)dx, \\ XIX = & 2 \int_0^1 \theta^T(t, x)R[\gamma]F_7[\gamma, \gamma_x, \zeta, \zeta_x, \theta]dx + 2 \int_0^1 \theta^T(t, x)R[\gamma]F_8[\gamma, \zeta, \theta]dx, \end{aligned}$$

$$XX = 2 \int_0^1 \theta^T(t, x) R[\gamma] G(x) \theta(t, 0) dx.$$

Let us first look at the second and the last term of (4.56) (i.e. XVII and XX), by some straight computations, one gets

$$\begin{aligned} XVII + XX &\leq -\theta_-(t, 0)^T \left(B - \tilde{S} - M\mu e^\delta \mathcal{C} - K_8 \|\gamma\|_\infty I_m \right) \theta_-(t, 0) \\ &\quad + \int_0^1 e^{-\delta x} \theta_+(t, x)^T \theta_+(t, x) dx + mM\mu \int_0^1 e^{\delta x} \theta_-(t, x)^T B \theta_-(t, x) dx \\ &\quad + K_9 \|\gamma\|_\infty V_3. \end{aligned} \quad (4.57)$$

Then by the same procedures in [8], we have the following

Proposition 4.3 *For any given $\lambda_3 > 0$, there exists $\delta_3 > 0$ and positive constants K_{10} , K_{11} , K_{12} , K_{13} and K_{14} , such that*

$$\dot{V}_3 \leq -\lambda_3 V_3 + K_{10} \|\gamma\|_\infty V_3 + K_{11} V_3 V_2^{\frac{1}{2}} + K_{12} V_2 V_3^{\frac{1}{2}} + K_{13} V_3^{\frac{3}{2}} + K_{14} \|\zeta\|_\infty^3, \quad (4.58)$$

provided that $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta_3$.

4.5 Proof of the H^2 stability for γ

Denote $W = V_1 + V_2 + V_3$, by Proposition 4.1, 4.2 and 4.3 as well as Lemma B.7, one can show that for any given $\lambda > 0$, there exists $\delta > 0$ and $K_{15} > 0$, such that

$$\dot{W} \leq -\lambda W + K_{15} W^{\frac{3}{2}}, \quad (4.59)$$

provided that $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta$. This concludes the whole proof of Theorem 1.1. \blacksquare

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Appendix A

In this section, we will show the well-posedness and piecewise smoothness of the Kernel K and L which are given by the following Theorems.

Theorem A.1 *Let $N \in \mathbb{N}^+$. Under the assumption that $\sigma_{ij} \in C^N[0, 1]$, $\lambda_i \in C^N[0, 1]$ ($i, j = 1, \dots, n$), there exists a unique piecewise $C^N(\mathcal{T})$ solution K to the hyperbolic system (2.45)-(2.49). Moreover, if the C^{N-1} compatibility conditions at the point $(x, \xi) = (1, 1)$ are satisfied, then $K(\cdot, \cdot) \in C^{N-1}(0, 1)$, $K(\cdot, 0) \in C^{N-1}(0, 1)$ with bounded C^{N-1} norm.*

Proof. We divided the proof into two parts. For the first part, we prove the regularity of the kernels. For this, we only prove the case $N = 1$. For $N \geq 1$, the results can be obtained by induction. In the case $N = 1$, one can, in fact, refer [12] and Remark A.1 to find there exists a piecewise C^0 kernel K for the boundary problem (2.45)-(2.49), where though only constant coupling coefficients and transport velocities are considered. However, the method in [12] straightforwardly extends to spatially varying coefficients with more involved technical developments. Next, we will

improve the regularity of K . Let $\mathcal{H}_{ij} = \partial_x K_{ij}(x, \xi)$ and $\mathcal{Y}_{ij} = \partial_\xi K_{ij}(x, \xi)$. By differentiating with respect to x in (2.45), one can show that

$$\lambda_i(x) \partial_x \mathcal{H}_{ij}(x, \xi) + \lambda_j(\xi) \partial_\xi \mathcal{H}_{ij}(x, \xi) = - \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj} \lambda_j'(\xi)) \mathcal{H}_{ik}(x, \xi) - \lambda_i'(x) \mathcal{H}_{ij}(x, \xi). \quad (\text{A.1})$$

Differentiating the boundary conditions in (2.46) and (2.47), we have

$$\mathcal{H}_{ij}(x, x) + \mathcal{Y}_{ij}(x, x) = k_{ij}'(x) \quad \text{for } 1 \leq i, j \leq n (i \neq j), \quad (\text{A.2})$$

$$\mathcal{H}_{ij}(x, 0) = - \frac{1}{\lambda_j(0)} \sum_{k=1}^{n-m} \lambda_{m+k}(0) \mathcal{H}_{i, m+k}(x, 0) q_{k,j} \quad \text{for } 1 \leq i \leq j \leq m. \quad (\text{A.3})$$

Next, differentiating the boundary conditions in (2.48)–(2.49), we have

$$\mathcal{Y}_{ij}(1, \xi) = \dot{k}_{ij}^{(1)}(\xi), \quad \text{for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n \quad (\text{A.4})$$

and the boundary conditions for $\mathcal{H}_{ij}(n \geq i \geq j \geq m+1)$ on $\xi = 0$:

$$\mathcal{H}_{ij}(x, 0) = \dot{k}_{ij}^{(2)}(x), \quad \text{for } m+1 \leq j \leq i \leq n. \quad (\text{A.5})$$

In view of the equations (2.45), it is easy to see that

$$\lambda_i(x) \mathcal{H}_{ij}(x, x) + \lambda_j(x) \mathcal{Y}_{ij}(x, x) = - \sum_{k=1}^n (\sigma_{kj}(x) + \delta_{kj} \lambda_j'(x)) K_{ik}(x, x) \quad (\text{A.6})$$

$$\lambda_i(1) \mathcal{H}_{ij}(1, \xi) + \lambda_j(\xi) \mathcal{Y}_{ij}(1, \xi) = - \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj} \lambda_j'(\xi)) K_{ik}(1, \xi) \quad (\text{A.7})$$

Combining (A.2) and (A.6), we have

$$\mathcal{H}_{ij}(x, x) = \frac{\lambda_j(x) k_{ij}'(x) + \sum_{k=1}^n (\sigma_{kj}(x) + \delta_{kj} \lambda_j'(x)) K_{ik}(x, x)}{\lambda_j(x) - \lambda_i(x)}, \quad \text{for } 1 \leq i, j \leq n (i \neq j). \quad (\text{A.8})$$

Similarly, plugging (A.7) into (A.4), one immediately obtains, for $1 \leq j < i \leq m \cup m+1 \leq i < j \leq n$, we have

$$\mathcal{H}_{ij}(1, \xi) = - \frac{1}{\lambda_i(1)} \left(\sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj} \lambda_j'(\xi)) K_{ik}(1, \xi) + \lambda_j(\xi) \dot{k}_{ij}^{(1)}(\xi) \right), \quad (\text{A.9})$$

which are piecewise $C^0(0, 1)$ function. By the theory in [12], we can prove that there exists a unique piecewise $\mathcal{H} \in C^0(\mathcal{T})$ for the boundary value problem (A.1), (A.3), (A.5) and (A.8)–(A.9). Noting the equations (2.45), we know that \mathcal{Y} shares the same regularity as \mathcal{H} .

Next, we prove the regularity of $K(\cdot, \cdot)$. Obviously, for $N = 1$, by the theory in [12] and Remark A.1, one can prove that both $K(\cdot, \cdot)$ and $K(\cdot, 0) \in C^0(0, 1)$ with bounded C^0 norm, provided that the C^0 compatibility conditions (2.50) are satisfied at the the point $(x, \xi) = (1, 1)$. Next, we prove the case $N = 2$. Taking an ξ -derivative in (2.45) yields

$$\begin{aligned} \lambda_i(x) \partial_x \mathcal{Y}_{ij}(x, \xi) + \lambda_j(\xi) \partial_\xi \mathcal{Y}_{ij}(x, \xi) = & - \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj} \lambda_j'(\xi)) \mathcal{Y}_{ik}(x, \xi) - \lambda_j'(\xi) \mathcal{Y}_{ij}(x, \xi) \\ & - \sum_{k=1}^n (\sigma_{kj}'(\xi) + \delta_{kj} \lambda_j''(\xi)) \mathcal{K}_{ik}(x, \xi) \end{aligned} \quad (\text{A.10})$$

Combining (A.2) and (A.6), we have

$$\mathcal{Y}_{ij}(x, x) = \frac{\lambda_i(x)k'_{ij}(x) + \sum_{k=1}^n (\sigma_{kj}(x) + \delta_{kj}\lambda'_j(x))K_{ik}(x, x)}{\lambda_i(x) - \lambda_j(x)}, \quad \text{for } 1 \leq i, j \leq n (i \neq j). \quad (\text{A.11})$$

Since

$$\lambda_i(x)\mathcal{H}_{ij}(x, 0) + \lambda_j(0)\mathcal{Y}_{ij}(x, 0) = - \sum_{k=1}^n (\sigma_{kj}(0) + \delta_{kj}\lambda'_j(0))K_{ik}(x, 0) \quad (\text{A.12})$$

Plugging (A.3) and (A.5), respectively, one obtains

$$\mathcal{Y}_{ij}(x, 0) = -\frac{1}{\lambda_j(0)} \left(\lambda_i(x)k^{(2)}_{ij}(x) + \sum_{k=1}^n (\sigma_{kj}(0) + \delta_{kj}\lambda'_j(0))K_{ik}(x, 0) \right), \quad \text{for } m+1 \leq j \leq i \leq n \quad (\text{A.13})$$

and

$$\begin{aligned} \mathcal{Y}_{ij}(x, 0) = & -\frac{1}{\lambda_j(0)} \sum_{k=1}^n (\sigma_{kj}(0) + \delta_{kj}\lambda'_j(0))K_{ik}(x, 0) + \frac{1}{\lambda_j^2(0)} \sum_{k=1}^{n-m} \lambda_{m+k}^2(0)q_{k,j}\mathcal{Y}_{i,m+k}(x, 0) \\ & + \frac{1}{\lambda_j^2(0)} \sum_{k=1}^{n-m} \sum_{s=1}^n \lambda_{m+k}(0)q_{k,j}(\sigma_{s,m+k}(0) + \delta_{s,m+k}\lambda'_{m+k}(0))K_{is}(x, 0), \quad \text{for } 1 \leq i \leq j \leq m. \end{aligned} \quad (\text{A.14})$$

Noting (A.4), (A.11), (A.13) and $K(\cdot, 0) \in C^0$, we know that $\mathcal{Y}_{ij}(\cdot, \cdot) \in C^0(0, 1) (i \neq j)$. $\mathcal{Y}_{ij}(1, \cdot) \in C^0(0, 1)$ (for $1 \leq j < i \leq m \cup m+1 \leq i < j \leq n$) and $\mathcal{Y}_{ij}(\cdot, 0) \in C^0(0, 1)$ (for $m+1 \leq j \leq i \leq n$). By the C^1 compatibility conditions (2.51) at the point $(x, \xi) = (1, 1)$ and using the theory in [12] and Remark A.1, we can prove that there exists a unique piecewise C^0 function $\mathcal{Y} = \mathcal{Y}(x, \xi)$ for the boundary value problem (A.10), (A.11), (A.13), (A.4) and (A.14), which satisfies $\mathcal{Y}(\cdot, \cdot)$, $\mathcal{Y}(\cdot, 0) \in C^0(0, 1)$. Noting (A.12) and (A.6), we know that $\mathcal{H}(\cdot, \cdot)$, $\mathcal{H}(\cdot, 0) \in C^0(0, 1)$. This finishes the proof. \blacksquare

Remark A.1 *It is worthy of mentioning that in [12], we only prove $K \in L^\infty(\mathcal{T})$ and do not clarify the regularity of the kernel because of brevity purposes. However, with the same procedure in [8, Section A.3] and [9], one can prove that K is a piecewise C^0 function with $K(\cdot, \cdot)$, $K(\cdot, 0) \in C^0(0, 1)$ and $K(1, \cdot)$ being a function of piecewise $C^0(0, 1)$ for the boundary problem (2.45)-(2.49), provided $\sigma_{ij} \in C^0[0, 1]$, $\lambda_i \in C^1[0, 1] (i, j = 1, \dots, n)$ and the C^0 compatibility conditions (2.50) are satisfied at the point $(x, \xi) = (1, 1)$.*

Theorem A.2 *Under the assumptions of Theorem A.1, For any $N \in \mathbb{N}$, there exists a unique piecewise $C^N(\mathcal{T})$ kernel L to the inverse transformation (2.52). Moreover, $L(x, x)$, $L(x, 0) \in C^{N-1}(0, 1)$.*

Proof. Substituting (2.40) for (2.52), it is easy to see that L is the solution of the following Volterra equations

$$L(x, \xi) = K(x, \xi) + \int_{\xi}^x K(x, s)L(s, \xi)ds, \quad (\text{A.15})$$

which yields that

$$L(x, x) = K(x, x) \in C^{N-1}(0, 1). \quad (\text{A.16})$$

Noting (2.43), we have

$$\Sigma(x) + L(x, x)\Lambda(x) - \Lambda(x)L(x, x) = 0. \quad (\text{A.17})$$

Next, Taking a partial derivative in x and ξ in (A.15), respectively, one obtains

$$L_x(x, \xi) = K_x(x, \xi) + K(x, x)L(x, \xi) + \int_{\xi}^x K_x(x, \xi)L(s, \xi)ds, \quad (\text{A.18})$$

$$L_{\xi}(x, \xi) = K_{\xi}(x, \xi) - K(x, \xi)L(\xi, \xi) + \int_{\xi}^x K(x, s)L_{\xi}(s, \xi)ds. \quad (\text{A.19})$$

Substituting (A.18) and (A.19) for (2.42) and using integration by parts, one has

$$\Lambda(x)L_x(x, \xi) + L_{\xi}(x, \xi)\Lambda(\xi) = (\Sigma(x) - \Lambda_{\xi}(\xi))L(x, \xi) \quad (\text{A.20})$$

Again by (A.15), we have

$$L(x, 0) = K(x, 0) + \int_0^x K(x, s)L(s, 0)ds. \quad (\text{A.21})$$

Since both $K(x, 0)$ and $K(x, x)$ are C^{N-1} continuous functions, by a suitable iteration procedure (see [17, Theorem 3.2, Pages 32–34]), it easy to see that there exists $L(x, 0) = l(x) \in C^{N-1}(0, 1)$ for the Volterra equation of the second kind (A.21).

On the other hand, substituting (2.52) for (2.40), one gets

$$L(x, \xi) = K(x, \xi) + \int_{\xi}^x L(x, s)K(s, \xi)ds, \quad (\text{A.22})$$

then

$$L(1, \xi) = K(1, \xi) + \int_{\xi}^1 L(1, s)K(s, \xi)ds. \quad (\text{A.23})$$

With the same argument above, we can see that $L_{ij}(1, \xi) = \tilde{l}_{ij}(\xi)$ ($m \geq i > j \geq 1, n \geq j > i \geq m+1$) on $x = 1$ are functions of piecewise $C^N(0, 1)$. Then, For the boundary problem (A.20) with the boundary conditions (A.17), and

$$L_{ij}(1, \xi) = \tilde{l}_{ij}(\xi), \text{ for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n \quad (\text{A.24})$$

$$L_{ij}(x, 0) = l_{ij}(x), \text{ for } 1 \leq i \leq j \leq m \cup m+1 \leq j \leq i \leq n. \quad (\text{A.25})$$

by Theorem A.1, one immediately gets Theorem A.2. ■

Appendix B

In this appendix, we first sketch out four useful lemmas (the details can be found in [8]).

Lemma B.1 *There exists a positive real number c_1 , such that*

$$|\mathcal{K}[\gamma]| + |\mathcal{L}[\gamma]| + |\mathcal{K}_1[\gamma]| + |\mathcal{K}_2[\gamma]| + |\mathcal{L}_1[\gamma]| \leq c_1(|\gamma| + \|\gamma\|_{L^1}). \quad (\text{B.1})$$

Lemma B.2 *Suppose $\|\gamma\|_{\infty}$ is suitable small, one can see that*

$$|F_1| \leq c_2(|\gamma| + \|\gamma\|_{L^1}), \quad (\text{B.2})$$

$$|F_2| \leq c_3(|\gamma|^2 + \|\gamma\|_{L^1}^2), \quad (\text{B.3})$$

$$|F_3| \leq c_4(|\gamma| + \|\gamma\|_{L^1})(\|\gamma_x\|_{L^2} + |\gamma_x|), \quad (\text{B.4})$$

$$|F_4| \leq c_5(|\gamma|^2 + \|\gamma\|_{L^1}^2). \quad (\text{B.5})$$

Lemma B.3

$$|F_{11}| \leq c_6(|\zeta| + \|\zeta\|_{L^1}), \quad (\text{B.6})$$

$$|F_{12}| \leq c_7(|\gamma_x| + |\gamma| + \|\gamma\|_{L^1}), \quad (\text{B.7})$$

$$|F_{21}| \leq c_8(|\gamma| + \|\gamma\|_{L^1})(|\zeta| + \|\zeta\|_{L^1}), \quad (\text{B.8})$$

$$|F_5| \leq c_9(|\zeta| + \|\zeta\|_{L^2})(|\gamma| + \|\gamma\|_{L^2}) + c_{10}(|\zeta| + \|\zeta\|_{L^2})(|\gamma_x| + \|\gamma_x\|_{L^2}) + c_{11}|\gamma(0)||\zeta(0)|, \quad (\text{B.9})$$

$$|F_6| \leq c_{12}(|\gamma| + \|\gamma\|_{L^2})(|\zeta| + \|\zeta\|_{L^2}). \quad (\text{B.10})$$

Lemma B.4

$$|F_{13}| \leq c_{13}(|\zeta|^2 + \|\zeta\|_{L^2}^2) + c_{14}(|\theta| + \|\theta\|_{L^1}), \quad (\text{B.11})$$

$$|F_{14}| \leq c_{14}(|\zeta| + \|\zeta\|_{L^1})(1 + |\gamma_x| + |\gamma| + \|\gamma\|_{L^1}) + c_{15}(|\zeta| + |\zeta_x| + \|\zeta\|_{L^1}), \quad (\text{B.12})$$

$$|F_{22}| \leq c_{16}(|\gamma| + \|\gamma\|_{L^1})(|\theta| + \|\theta\|_{L^1}) + c_{17}(|\zeta|^2 + \|\zeta\|_{L^2}^2), \quad (\text{B.13})$$

$$\begin{aligned} |F_7| &\leq c_{18}(|\zeta|^2 + \|\zeta\|_{L^2}^2)(1 + |\gamma| + \|\gamma_x\|) \\ &\quad + c_{19}(|\zeta| + \|\zeta\|_{L^2})(|\zeta_x| + \|\zeta\|_{L^2}) \end{aligned} \quad (\text{B.14})$$

$$\begin{aligned} &+ c_{20}(|\gamma| + \|\gamma\|_{L^2} + |\gamma_x|)(|\theta| + \|\theta\|_{L^2}) \\ &+ c_{21}(|\zeta(0)|^2 + |\gamma(0)||\theta(0)|), \end{aligned}$$

$$|F_8| \leq c_{22}(|\zeta|^2 + \|\zeta\|_{L^2}^2)(1 + \|\gamma\|_\infty) + c_{23}(|\gamma| + \|\gamma\|_{L^2})(|\theta| + \|\theta\|_{L^2}). \quad (\text{B.15})$$

Next, we show the following proposition which is also mentioned in [8], however here more technical developments are involved.

Proposition B.1 *There exists $\delta > 0$ such that for any $|\gamma| + |\zeta| \leq \delta$, one has*

$$\|\theta\|_\infty \leq C_1(\|\gamma_{xx}\|_\infty + \|\gamma_x\|_\infty + \|\gamma\|_\infty), \quad (\text{B.16})$$

$$\|\theta\|_{L^2} \leq C_2(\|\gamma_{xx}\|_{L^2} + \|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \quad (\text{B.17})$$

$$\|\gamma_{xx}\|_\infty \leq C_3(\|\theta\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \quad (\text{B.18})$$

$$\|\gamma_{xx}\|_{L^2} \leq C_4(\|\theta\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \quad (\text{B.19})$$

where C_1, C_2, C_3 and C_4 are positive constants.

Proof. We prove the next three lemmas to get Proposition B.1.

Lemma B.5 *There exists δ such that, if $|\gamma| \leq \delta$, then the following inequalities hold:*

$$\|\zeta\|_\infty \leq c_1(\|\gamma_x\|_\infty + \|\gamma\|_\infty) \quad (\text{B.20})$$

$$\|\zeta\|_{L^2} \leq c_2(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \quad (\text{B.21})$$

$$\|\gamma_x\|_\infty \leq c_3(\|\zeta\|_\infty + \|\gamma\|_\infty), \quad (\text{B.22})$$

$$\|\gamma_x\|_{L^2} \leq c_4(\|\zeta\|_{L^2} + \|\gamma\|_{L^2}) \quad (\text{B.23})$$

Proof. Noting (4.12), one can easily see that

$$\zeta(t, x) + \Lambda(x)\gamma_x(t, x) - G(x)\gamma(t, 0) = F_3[\gamma, \gamma_x] + F_4[\gamma]. \quad (\text{B.24})$$

The difference between our proof and the proof in [8, Lemma B.6] is the appearance of the term $G(x)\gamma(t, 0)$ in (B.24). Noting (2.44) and Theorem A.1, we have $G(\cdot) \in C^1(0, 1)$ with bounded C^1 norm. Then since one can show that

$$\|G(\cdot)\gamma(t, 0)\|_{L^2} \leq c_5\|G(\cdot)\gamma(t, 0)\|_\infty \leq c_6\|\gamma\|_\infty \leq c_7(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \quad (\text{B.25})$$

which yields, by the same argument in [8, Lemma B.6], (B.20)-(B.22).

On the other hand, by the special structure of $G(x)$, we have

$$\|\partial_x \gamma_1\|_{L^2} \leq c_8(\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \quad (\text{B.26})$$

$$\|\partial_x \gamma_2\|_{L^2} \leq c_9(\|\zeta\|_{L^2} + \|\gamma_1\|_\infty + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \quad (\text{B.27})$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\|\partial_x \gamma_m\|_{L^2} \leq c_{m+7}(\|\zeta\|_{L^2} + \sum_{r=1}^{m-1} \|\gamma_r\|_\infty + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \quad (\text{B.28})$$

$$\|\partial_x \gamma_s\|_{L^2} \leq c_{s+7}(\|\zeta\|_{L^2} + \sum_{r=1}^m \|\gamma_r\|_\infty + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \quad (\text{B.29})$$

in which $s = m+1, \dots, n$. Noting the classical Sobolev's inequality

$$\|\gamma\|_{L^\infty} \leq \tilde{C}(\|\gamma\|_{L^2} + \|\gamma_x\|_{L^2}) \leq \tilde{C}\|\gamma\|_{H^1}, \quad (\text{B.30})$$

one gets that

$$\|\partial_x \gamma_1\|_{L^2} \leq C_1(\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \quad (\text{B.31})$$

$$\|\partial_x \gamma_2\|_{L^2} \leq C_2(\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \|\partial_x \gamma_1\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \quad (\text{B.32})$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\|\partial_x \gamma_m\|_{L^2} \leq C_m(\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \sum_{r=1}^{m-1} \|\gamma_r\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \quad (\text{B.33})$$

$$\|\partial_x \gamma_s\|_{L^2} \leq C_s(\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \sum_{r=1}^m \|\gamma_r\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_{L^\infty} + \|\gamma\|_{L^2}\|\gamma\|_\infty), \quad (\text{B.34})$$

where $s = m+1, \dots, n$. Then, we can easily obtain by induction that

$$\|\gamma_x\|_{L^2} \leq \tilde{c}_1(\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}), \quad (\text{B.35})$$

which concludes (B.23), under the assumption that $\|\gamma\|_\infty$ is small enough. \blacksquare

Combining the same technical approach as in [8, Lemma B.7 and Lemma B.8] and an analogous argument used in the proof of Lemma B.5 and noting $G \in C^1$, the details of which we omit, one can show the next two lemmas.

Lemma B.6 *There exists δ such that, if $\|\gamma\|_\infty \leq \delta$, then the following inequalities hold:*

$$\|\gamma_{xx}\|_\infty \leq c_1(\|\zeta_x\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \quad (\text{B.36})$$

$$\|\gamma_{xx}\|_{L^2} \leq c_2(\|\zeta_x\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \quad (\text{B.37})$$

$$\|\zeta_x\|_\infty \leq c_3(\|\gamma_{xx}\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \quad (\text{B.38})$$

$$\|\zeta_x\|_{L^2} \leq c_4(\|\gamma_{xx}\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \quad (\text{B.39})$$

where c_1, c_2, c_3 and c_4 are positive constants.

and

Lemma B.7 *There exists δ such that, if $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta$, then the following inequalities hold:*

$$\|\theta\|_\infty \leq c_1(\|\zeta_x\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \quad (\text{B.40})$$

$$\|\theta\|_{L^2} \leq c_2(\|\zeta_x\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \quad (\text{B.41})$$

$$\|\zeta_x\|_{\infty} \leq c_3(\|\theta\|_{\infty} + \|\zeta\|_{\infty} + \|\gamma\|_{\infty}), \quad (\text{B.42})$$

$$\|\zeta_x\|_{L^2} \leq c_4(\|\theta\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \quad (\text{B.43})$$

where c_1, c_2, c_3 and c_4 are positive constants.

The above three Lemma B.5–B.7 immediately yield Proposition B.1. ■

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